Physics of climbing ropes: impact forces, fall factors and rope drag

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Introduction

In a previous work [1] the behaviour of a climbing rope with inner viscous friction was presented. This type of friction dominates in the UIAA heavy fall case, where the fall factor is near 2.

In this work we extend the description and show how a climbing rope behaves under falls with smaller fall factors than under UIAA test fall conditions. In the case of one or more protection points (usually bolts with quickdraws) between leader and belayer one has to take into account the so-called dry friction between rope and the protection points.

It will be shown that dry friction leads to the same form of equations and thus the same expressions for dynamic elongation and impact force as in the case without dry friction, if the elastic modulus is redefined properly.

Furthermore the rope drag is calculated depending on the number of protection points and the angle deviations of the rope at these points.

Other work on this subject can be found in [2] with a numerical simulation approach. [3] presents experimental data of drop tests. [4] and [5] discuss climbing rope properties on the basis of the harmonic oscillator model and use some heuristic and energy balance considerations to describe dry friction.

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[1] U. Leuthäusser, Viscoelastic Theory of Climbing Ropes, *www.sigmadewe.com/fileadmin/user_upload/pdf-Dateien/Physics_of_climbing_ropes.pdf*
[2] M. Pavier, Experimental and theoretical simulations of climbing falls, Sports Engineering (1998) **1**, 79-91
[3] J. Marc Beverly, Stephen W. Attaway, Measurement of Dynamic Rope System Stiffness in a Sequential Failure for Lead Climbing Falls, *www.mra.org/drupal2/sites/default/files/.../Beverly_Sequential_Falls2.pdf*
[4] M. Lutzenberger, Seiltechniken für Bergführer, *www.swissgeocache.ch/forum/media/Seilkunde.pdf*
[5] S. W. Attaway, Rope System Analysis, *http://lamountaineers.org/xRopes.pdf*

1. Friction between rope and protection points

We consider the situation depicted in the figure below. The leading climber takes a fall of a distance $2l_n$ above the last protection P_{n-1}. At the end of the fall when the rope begins to stretch he has a velocity $v_0 = \sqrt{4gl_n}$ and his location is called x_n. The rope responds to the fall with elongations x_i at P_i (i=1, ..., n-1). The rope segments and their spring constants between P_{i-1} and P_i are denoted as l_i and k_i. Together with the friction constant μ the angle α_i at P_i determines the friction force at this point.

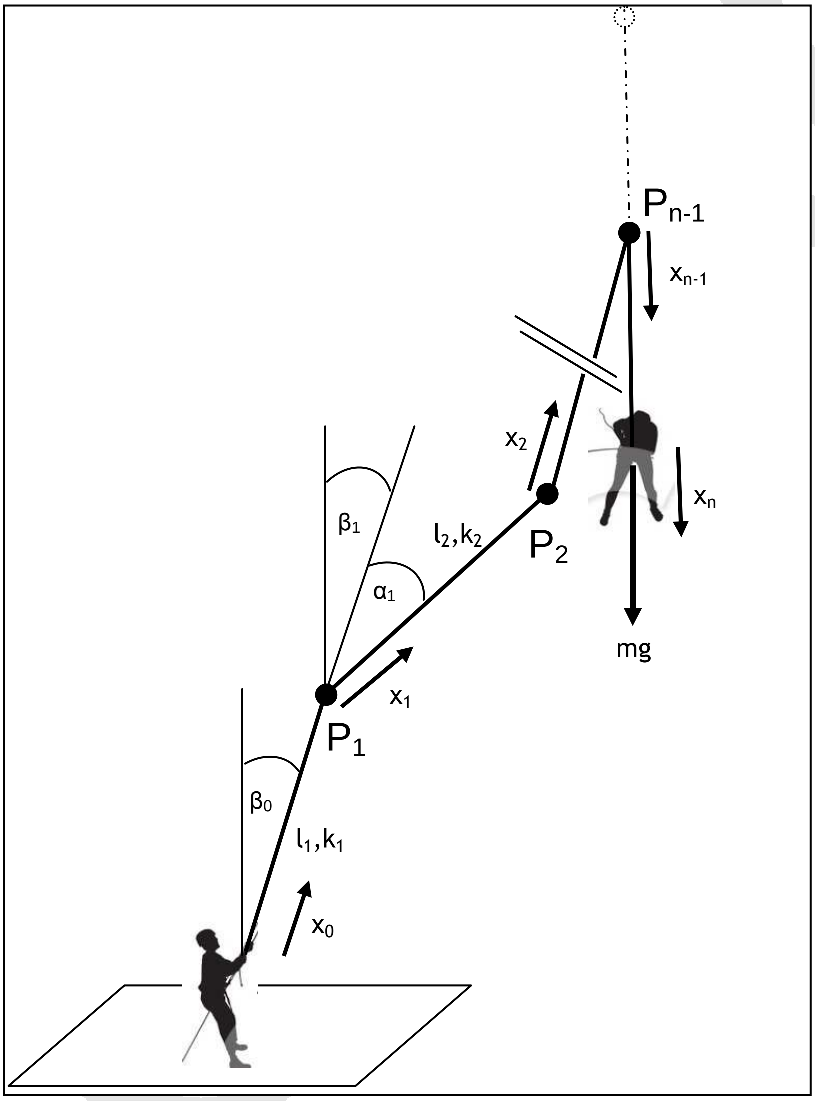

Fig.1

Viscous damping excluded

To keep things relatively simple, we first omit viscous friction and start with the Lagrange function

$$L = \frac{m}{2}\dot{x}_n^2 - \left[\frac{1}{2}k_1(x_1 - x_0)^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \ldots + \frac{1}{2}k_n(x_n - x_{n-1})^2 - mgx_n\right] \qquad (1)$$

The coordinate x_0 will be used as a control variable in a next publication. For a static belayer x_0 is zero.

The Lagrange equations for non conservative systems are given by

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = F_i \qquad (2)$$

with the dissipative forces

$$F_i = k_i(x_{i+1} - x_i)(1 - e^{-\mu\alpha_i}) \qquad (3)$$

using the equation of Euler-Eytelwein (see Figure 2). Their dependency on the direction of motion is omitted here, thus they are only valid for short times including, however, the times of maximum elongation and acceleration. A more detailed discussion of dry friction can be found in appendix A.

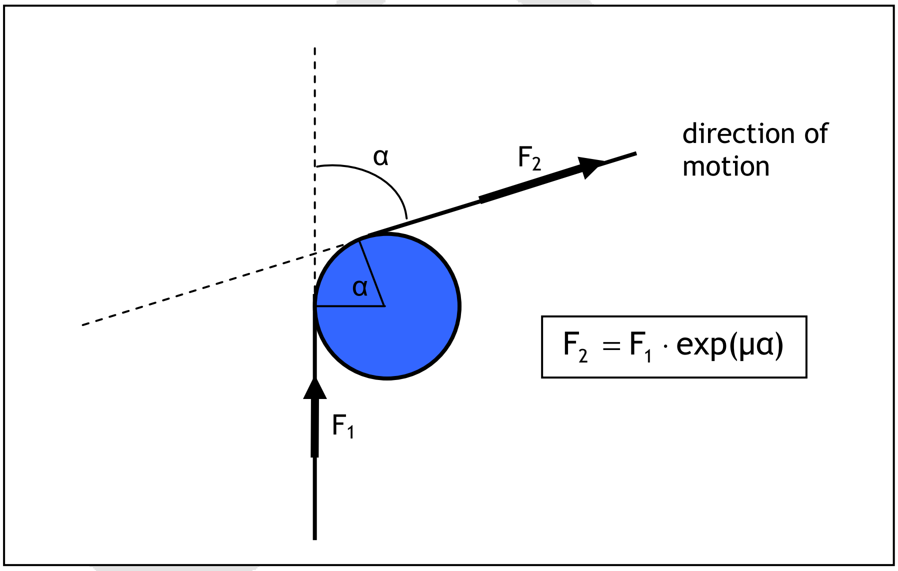

Fig.2. When a force F_2 pulls a rope over a curved surface with friction μ, the force F_1 on the opposite side is reduced and given by $F_1 = F_2 \exp(-\mu\alpha)$. This force depends on the contact angle α between rope and surface (formula of Euler-Eytelwein), but not on the curvature of the surface.

From (2) and (3) it follows immediately

$$
\begin{aligned}
&k_2(x_2 - x_1)e^{-\mu\alpha_1} - k_1(x_1 - x_0) = 0\\
&k_3(x_3 - x_2)e^{-\mu\alpha_2} - k_2(x_2 - x_1) = 0\\
&\vdots\\
&k_i(x_i - x_{i-1})e^{-\mu\alpha_{i-1}} - k_{i-1}(x_{i-1} - x_{i-2}) = 0\\
&\vdots\\
&k_n(x_n - x_{n-1})e^{-\mu\alpha_{n-1}} - k_{n-1}(x_{n-1} - x_{n-2}) = 0\\
&m\ddot{x}_n + k_n(x_n - x_{n-1}) = mg
\end{aligned}
\qquad (4)
$$

This system of equations can also be obtained without the formalism of Lagrange. The last equation is a Harmonic Oscillator equation for the mass m. No other masses are involved, thus the equations represent the balance of forces at the protection points considering friction.

It is possible to get an equation for the elongation x_n of the form

$$m\ddot{x}_n + k_{eff}(x_n - x_0) = mg \qquad (5)$$

from the system (4). In order to eliminate all the intermediate x_i, we solve the first equation for x_1, substitute it in the second equation which leads to

$$k_3(x_3 - x_2) = \frac{x_3 - x_0}{\dfrac{1}{k_3} + \dfrac{1}{e_2 k_2} + \dfrac{1}{e_1 e_2 k_1}}$$

with the abbreviation $e_i = \exp(\mu\alpha_i)$. Continuing this procedure one obtains

$$k_n(x_n - x_{n-1}) = \frac{x_n - x_0}{\dfrac{1}{k_n} + \dfrac{1}{e_{n-1}k_{n-1}} + \ldots + \dfrac{1}{e_1 e_2 \ldots e_{n-1} k_1}}$$

Comparison with equation (5) yields the effective spring constant

$$\frac{1}{k_{eff}} = \frac{1}{k_n} + \frac{1}{e_{n-1}k_{n-1}} + \ldots + \frac{1}{e_1 e_2 \ldots e_{n-1} k_1} \qquad (6)$$

The k_i depend on the lengths l_i:

$$k_i = \frac{E \cdot q}{l_i} \qquad (7)$$

where E is the elastic modulus and q the cross section of the rope. Substituting the k_i into equation (5), it reduces to

$$k_{eff} = \frac{Eq}{l_n + e_{n-1}^{-1} l_{n-1} + \ldots + e_1^{-1} e_2^{-1} \ldots e_{n-1}^{-1} l_1} = \frac{Eq}{l_{eff}} \qquad (8)$$

with an effective rope length

$$l_{eff} = l_n + e_{n-1}^{-1} l_{n-1} + \ldots + e_1^{-1} e_2^{-1} \ldots e_{n-1}^{-1} l_1 < l$$

that is always smaller than l. The formula for k_{eff} is very intuitive and can almost be guessed: if there is no friction (all e_i=1), k_{eff} is given by $\frac{1}{k_{eff}} = \sum_{i=1}^{n} \frac{1}{k_i}$, the well known formula for springs in series. In this case we find the lowest possible value for k_{eff} = Eq/l, the spring constant of a rope with length l. In the opposite case $e_{n-1} \to \infty$ of infinite friction (i.e. the rope is pinned at the last protection point) only the last part of the rope acts as an oscillator and k_{eff} obtains its maximum possible value k_{eff} = Eq/l_n without any energy dissipation. In between, the spring constant k_{n-1} is replaced by a larger effective spring constant $e_{n-1} \cdot k_{n-1}$, k_{n-2} replaced by $e_{n-1} e_{n-2} \cdot k_{n-2}$, and so on. Thus dry friction always leads to a higher spring constant. Taking UIAA norm fall conditions, the spring constant is only slightly increased, because of the very small l_1 = 0.3m compared to the total l=2.6m

$$k_{eff} = \frac{e^{\mu\alpha} \cdot k_1 k_2}{e^{\mu\alpha} \cdot k_1 + k_2} \approx \frac{Eq}{l}\left(1 + \left(1 - e^{-\mu\alpha}\right)\frac{l_1}{l}\right) = \frac{Eq}{l}(1 + 0.06)$$

with μ = 0.25 and $\alpha = \pi$. Thus the error in neglecting the friction between rope and carabiner is about 6%.

In the limiting case of infinite friction $\mu \to \infty$ one gets $x_0 = x_1 = \ldots \; x_{n-1}$, which are not necessarily zero, because we have neglected the mass of the rope. Taking into account the rope mass the motion of x_1 is prevented, i.e. $x_1 = \ldots \; x_{n-1} = 0$. Because the rope is flexible, a rope feed from the belayer at x_0 leads only to a slack rope but not to a motion of x_1.

Taking the expression for the impact force on a rope in the Harmonic Oscillator approximation [1]

$$F^{max} = mg + m \cdot \max(|\ddot{x}_n|) = mg + \sqrt{2mghk_{eff} + m^2 g^2} = mg + \sqrt{2mgh\frac{Eq}{l_{eff}} + m^2 g^2} \qquad (9)$$

an effective fall factor can now be defined as

$$f_{eff} = f\frac{l}{l_{eff}} = \frac{h}{l_{eff}}$$

where f=h/l is the usual fall factor without friction. F^{max} with friction divided by F^{max} without friction (l_{eff}=l) scales approximately like $\sqrt{l/l_{eff}} \geq 1$.

The frictional force on the last protection point n-1 is given by

$$F_{n-1} = k_n(x_n - x_{n-1})(1 - e^{-\mu\alpha_{n-1}})$$

For its maximum value $F_{n-1}^{max} = F^{max}(1 - e^{-\mu\alpha_{n-1}})$ and with typical values for μ=1/4 and $\alpha_{n-1} \approx \pi$, we find $F_{n-1}^{max} = 0.544 \cdot F^{max}$.

The corresponding fall factor is given by $f_{eff} = \dfrac{2f}{(1 - e^{-\mu\alpha_{n-1}})f + 2e^{-\mu\alpha_{n-1}}}$.

For f=1 one finds f_{eff} = 1.374.

The maximum force on the belay F_B^{max} can be easily expressed by the impact force F^{max} by eliminating all $k_i(x_i - x_{i-1})$ in (4)

$$F_B^{max} = k_1 \max(x_1 - x_0) = F^{max} \prod_{i=1}^{n-1} e_i^{-1} = F^{max} \cdot \exp\left(-\mu \sum_{i=1}^{n-1} \alpha_i\right)$$

The sum of all α_i appear in the exponent, thus for high friction the exponential leads to a very small F_B^{max}: the impact force is distributed among the protection points and cannot propagate to the belayer.

The maximum force on the last protection point (LPP) is given by

$$F_{LPP}^{max} = 2F^{max} - F_{n-1}^{max} = F^{max}(1 + e^{-\mu\alpha_{n-1}}).$$

With the values above, we obtain $F_{LPP}^{max} = 1.46\, F^{max}$.
Although F^{max} of equation (9) is increased by dry friction, the factor $(1 + e^{-\mu\alpha_{n-1}})$ can overcompensate this effect and in total F_{LPP}^{max} is lower for moderate friction. If, however, inner viscous friction is taken into account, this effect disappears (see Fig.4).

Viscous damping included

Now we add the viscous damping of the rope. Because of the relative complicated calculations even for the Maxwell model, the work is done in the Appendix A and here only the result is presented.
For short times the equation of the rope, generalizing equation (5) is given by

$$m\ddot{x}_n + k(x_n - x_0) + m\frac{k}{\eta}(\dot{x}_n - v_0) = mg \qquad (10)$$

A derivation can be found in [1]. For η we have to use

$$\frac{1}{\eta} = \frac{1}{\eta_n} + \frac{1}{e_{n-1}\eta_{n-1}} + \ldots + \frac{1}{e_1 e_2 \ldots e_{n-1}\eta_1}$$

where the η_i are the viscosities in the i^{th} section of the rope. The total η has the same structure as k and also the same scaling behaviour with respect to l. Thus the expression k/η is length independent and is denoted by

$$\kappa = 2\frac{k}{\eta}.$$

In [1], equation (10) is solved and discussed in detail. Approximate expressions for the maximum dynamic elongation and the impact force are given by

$$F^{max} \cong m(g + \frac{2}{3}\kappa v_0) + m\sqrt{v_0^2 \cdot \omega^2 + (g + \frac{2}{3}\kappa v_0)^2} - 2m\kappa v_0 \qquad (11)$$

$$x^{max} \cong \frac{1}{\omega^2}\sqrt{v_0^2 \cdot \omega^2 + (g + \frac{2}{3}\kappa v_0)^2} + \frac{g + \frac{2}{3}\kappa v_0}{\omega^2} \qquad (12)$$

where we have to use

$$\omega = \sqrt{k/m} = \sqrt{\frac{Eq/m}{l_n + e_{n-1}^{-1} l_{n-1} + \ldots + e_1^{-1} e_2^{-1} \ldots e_{n-1}^{-1} l_1}} \quad \text{and}$$

$$v_0 = \sqrt{4 g l_n}.$$

Note the importance of equation (11) which gives the impact force for many fall situations with an arbitrary number of protection points and various vertical fall distances $2l_n$. Parameters are the rope material constants with typical values $Eq \sim 3.6 \cdot 10^4$ N and κ~3.

In Fig.3 the impact force F^{max} from equation (11) is plotted as a function of the dry friction parameter $1 - e^{-\mu\alpha}$ for various situations with only one protection point. In all cases the total rope length is l = 20m and the fall height $2l_2$ takes the values 5m (solid), 10m (dotted), 20m (dashdot). The red curves are F^{max} without viscous damping and rise with increasing dry friction. For infinite friction they end at the same value on the right side, because the fall factor is 2 for all three fall situations. The blue curves include viscous damping (κ = 3, a typical value, see [1]) which considerably reduces the impact forces. They don't end in the same point on the right side, because they are not solely dependent on the fall factor.

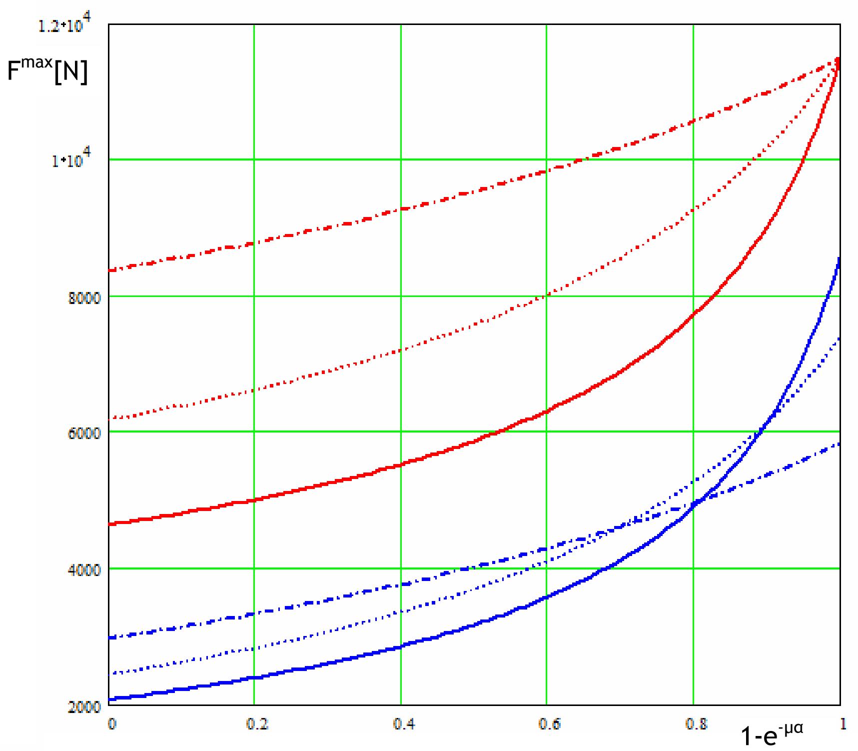

Fig. 3

Using the same rope length and fall heights, the force on the last protection point F_{LPP}^{max} is shown in Fig.4. The blue curves which include viscous damping show that dry friction has only little effect on F_{LPP}^{max} .

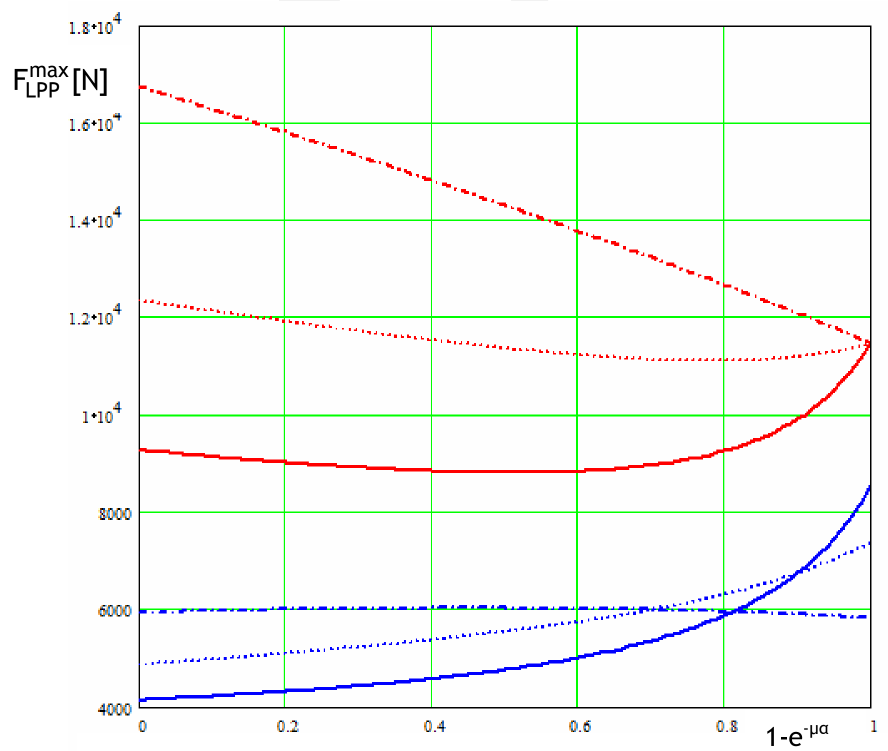

Fig.4

Finally the fall factor for an entire climbing route is shown in Fig.5. We take case 6 from Fig.6 of the next section. The length of each rope segment is l_i=2m. All e_i are the same as in Fig.6 except with a very high e_5=10 resulting in a high fall factor after the climber is beyond P_5. (blue: the fall factor without dry friction, red: with dry friction). Note the typical climbing situation which is most dangerous at the beginning of the climb before the first protection is reached. Clipping the first protection point the fall factor jumps to zero, then increases again after climbing away from the protection.

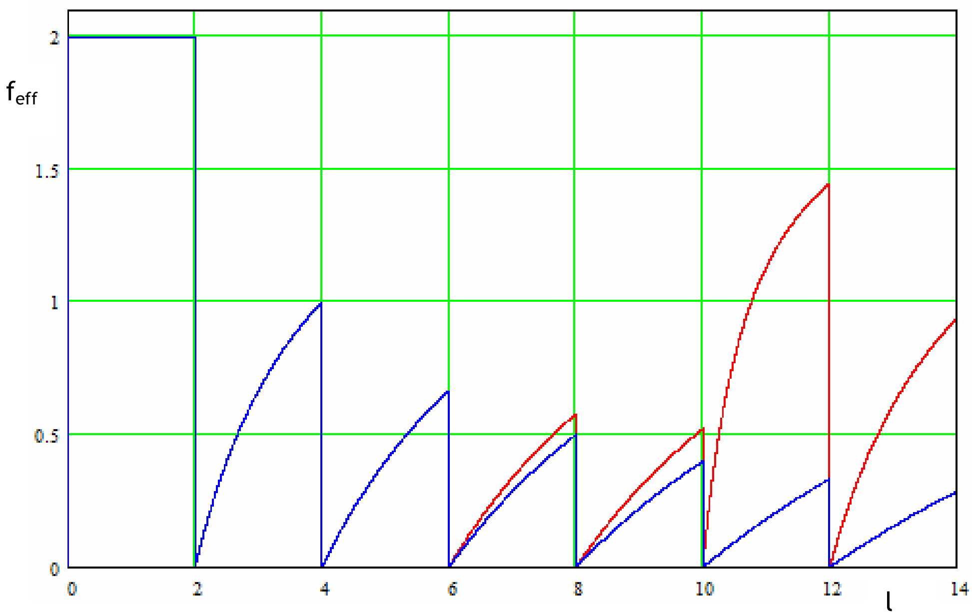

Fig. 5

2. Rope drag

It is also possible to calculate the rope drag, i.e. the friction of the rope plus its weight, the climber feels moving forward.
The force in front of P_1 (coming from below) is given by $\rho l_1 \cos(\beta_1) \cdot g$ (ρ is the specific weight of the rope), so that the force behind P_1 is given by

$$T_1 = \rho l_1 \cos(\beta_1) g e^{\mu\alpha_1}$$

using the equation of Euler-Eytelwein and which is larger than $\rho l_1 \cos(\beta_1) g$. The force behind P_2 is the sum of T_1 and the weight of the next line element l_2 multiplied by $e^{\mu\alpha_2}$ in order to overcome the friction at P_2:

$$T_2 = (T_1 + \rho l_2 \cos(\beta_2) g) e^{\mu\alpha_2}$$

Behind P_i we have

$$T_i = (T_{i-1} + \rho l_i \cos(\beta_i) g) e^{\mu\alpha_i} =$$
$$\rho l_i \cos(\beta_i) g e^{\mu\alpha_i} + \rho l_{i-1} \cos(\beta_{i-1}) g e^{\mu(\alpha_i+\alpha_{i-1})} + \ldots + \rho l_1 \cos(\beta_1) g e^{\mu(\alpha_i+\alpha_{i-1}+\ldots+\alpha_1)} \quad (13)$$

Finally one arrives at the last P_{n-1}. The minimal drag force F_D that the climber needs to move forward is now given by

$$F_D = T_n =$$
$$T_{n-1} + \rho l_n g = \rho g \left(l_n \cos(\beta_n) + l_{n-1} \cos(\beta_{n-1}) \cdot e_{n-1} + \ldots + l_1 \cos(\beta_1) e_1 e_2 \ldots e_{n-1} \right) \quad (14)$$

or in a more compact notation defining an effective mass of the rope

$$F_D = g m_{eff}^{rope} = g \rho \sum_{i=1}^{n} l_i \cos(\beta_i) \prod_{j=i}^{n-1} e_j$$

In the case of no friction, if all e_i's are one, F_D is simply the weight of the rope $\rho l g$ multiplied by an average cosine $< \cos(\beta) >= \frac{1}{l} \sum_{i=1}^{n} l_i \cos(\beta_i)$.
If the β_i, l_i and e_i are all constant, one obtains a drag force

$$F_D = g \rho l \cos(\beta) \frac{1}{n} \frac{e^{n\mu\alpha} - 1}{e^{\mu\alpha} - 1}$$

exponentially increasing with the total angle $n\alpha$. Under normal conditions when the α's are small, F_D is given by $F_D \cong g \rho l \cos(\beta) \left(1 + 1/2 \cdot \mu\alpha(n-1)\right)$. For $\alpha = \pi/10$, $n = 10$ and $\mu = 1/4$ the effective weight the climber has to pull increases about 35% compared to the case without friction.

The next figure shows a rope with 6 protection points e_1-e_6. The total angle deviation $4 \cdot \pi/4$ is the same in all 6 cases. In spite of the apparent equivalency, the rope drags are different. In case 1, only rope segment l_1 has to be pulled through e_1, e_2, e_3 and e_4 larger than 1. In case 6, however, the segments l_1-l_3 must be pulled through e_3-e_6. Thus the rope drag is larger in this case, in contrast to intuition.

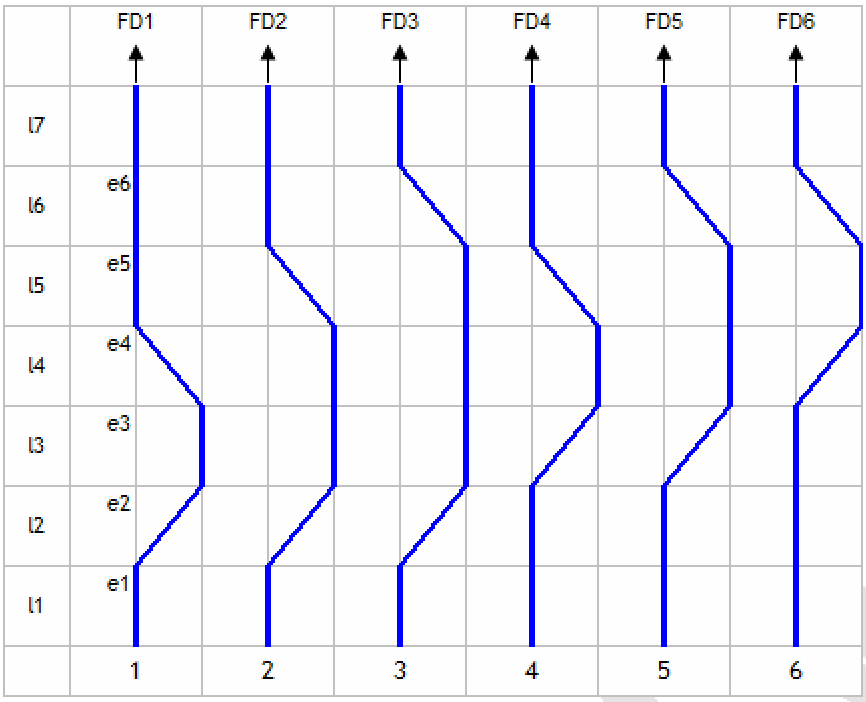

Fig.6: Six situations of rope drag with the same total friction angle. Case 1 has the lowest rope drag, the following are in ascending order ending with the highest rope drag for case 6. Taking for $e_i = \exp(\mu\pi/4) \approx 1.22$ one obtains in case 1 an effective mass $m_{eff}^{rope} = 1.38 m^{rope}$ and in case 6 $m_{eff}^{rope} = 1.73 m^{rope}$.

3. Conclusions

In this work we derived an expression for the impact force (equation 11) and maximum elongation (equation 12) including dry friction as well as internal viscous friction for all kinds of climbing situations: with arbitrary protection points, friction coefficients, angles between rope and protection points. It turned out that the original form of the equations is unchanged, if one redefines the spring constant of the rope by introducing an effective rope length, which leads to an effective fall factor. Because of the easy explicit expressions one can calculate at once the impact force for many climbing situations.

Dry friction leads first of all to a higher (stiffer) effective elastic modulus. Energy dissipation due to dry friction is smaller than viscous damping: the reason why a rope has almost no oscillation is viscous damping and not dry friction. In the limit of infinite dry friction there is only energy dissipation from viscous friction.

Furthermore we calculated the rope drag a climber has to overcome in order to move forward. Its only source is dry friction. It can also be expressed by an effective mass which is larger than the mass of the rope that has to be pulled by the climber. This effective mass depends exponentially on the sum of the angles of the direction changes the climber has made. “Early errors” not using longer runners to reduce the angles α at the first protection points are less severe than “later errors” which is in contrast to intuition.

Appendix A

In this appendix we discuss the equations (4) for n=2, i.e. for only one protection point P_1, in more detail. This special case is important, because the last protection has usually the largest friction ($\alpha = \pi$), and is therefore a limiting case of (4) when all α_i can be neglected except of the last one.
Assuming for the moment a small mass m_1 at P_1, one obtains from the Lagrangian

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 (x_2 - x_1)^2 + m_2 g x_2 \qquad \text{(A1)}$$

the following equations of motion

$$\begin{aligned} m_1 \ddot{x}_1 + k_1 x_1 - k_2 (x_2 - x_1) &= F_1 \\ m_2 \ddot{x}_2 + k_2 (x_2 - x_1) &= m_2 g \end{aligned} \qquad \text{(A2)}$$

The friction force F1 is given by

$$F_1 = -(1 - e^{-\alpha_1 \mu}) \cdot \operatorname{sgn}(\dot{x}_1) \cdot \max\left(\left| k_2 (x_2 - x_1) \right|, \left| k_1 x_1 \right| \right) \qquad \text{(A3)}$$

with the sign function defined as

$$\operatorname{sgn}(a) = \begin{cases} -1 & \text{if } a < 0 \\ 0 & \text{if } a = 0 \\ 1 & \text{if } a > 0 \end{cases}$$

The frictional force at P_1 always has the opposite sign of the velocity at P1, its magnitude is independent of the velocity, but depends on the maximum of the two forces acting on either side of P_1.
Equations (A2) must be solved with the initial conditions $\dot{x}_2(0) = v_0$, $\dot{x}_1(0) = 0$, $x_1(0) = 0$, $x_2(0) = 0$. The initial velocity v_0 after a fall of $2l_n$ is usually sufficiently large so that $m_2 g$ can be neglected.
Let us discuss the time interval until the rope attains its first zero crossing, beginning with the first half

1. $0 \le t \le \frac{T}{4} = \frac{1}{4} \frac{2\pi}{\omega} = \frac{\pi}{2} \sqrt{\frac{m_2}{k_2}}$

This time interval ends when the rope reaches its maximum elongation and its impact force and during this time interval the following relations are valid

$$k_2 (x_2 - x_1) > k_1 x_1 > 0 \quad \text{and} \quad v_2 > v_1 > 0$$

Taking F_1 from (A3) one gets

$$m_1\ddot{x}_1 + k_1x_1 - \frac{1}{e_1}\cdot k_2(x_2 - x_1) = 0$$
$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0 \tag{A4}$$

and for $m_1 \rightarrow 0$

$$x_1 = \frac{k_2}{e_1k_1 + k_2}x_2$$
$$m_2\ddot{x}_2 + \frac{k_1k_2e_1}{k_1e_1 + k_2}x_2 = 0 \tag{A5}$$

The second time interval ends when x_1 and x_2 have again their initial values:

2. $\frac{T}{4} < t \leq \frac{T}{2} = \pi\sqrt{\frac{m_2}{k_2}}$

At the beginning of this time interval the velocities are zero, but we still have $k_2(x_2 - x_1) > k_1x_1 > 0$. The total energy rate is given by

$$\frac{dE}{dt} = -v_1(t)\big(k_2(x_2(t) - x_1(t)) - k_1x_1(t)\big) \leq 0$$

and can never increase. This can only be satisfied if $v_1 \geq 0$. Thus, it follows $v_1 = 0$ at least as long as $k_2(x_2 - x_1) > k_1x_1$. x_1 is constant and there is no energy dissipation. That is very surprising and interesting. One could expect that the motion of x_1 begins again at the time corresponding to $k_2(x_2 - x_1) = k_1x_1$. Numeric integration of the equations of motion, however, show that the time interval with $v_1 = 0$ ends at a time t_1 which is somewhat longer than the time corresponding to $k_2(x_2 - x_1) = k_1x_1$. At the time t_1 the velocity v_1 immediately jumps to

$$v_1 = \frac{k_2}{e_1k_1 + k_2}v_2$$

valid until the zero crossing of the rope elongation x_2.

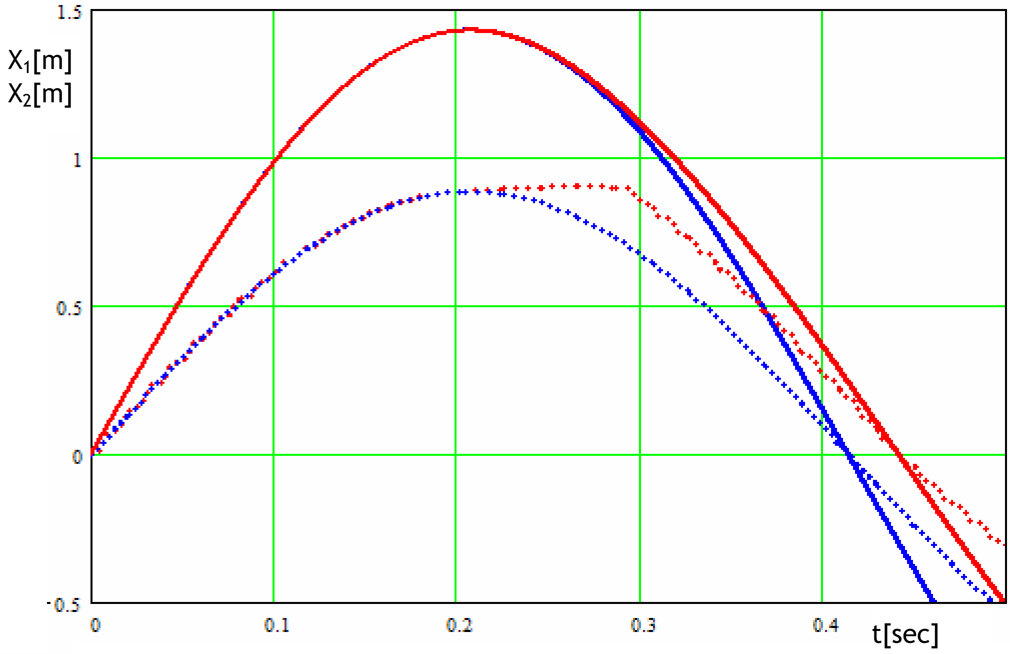

Fig. 7: The blue curves are the elongations x_1 (solid) and x_2 (dotted) from equations (A5) as a function of time. The red ones are calculated numerically. Equations (A5) are exact up to the maximum of x_1 and x_2.

A full discussion of dry friction is beyond the level of this paper and fortunately not particularly important here, because we are interested in the influence of dry friction on the maximum elongation and impact force which takes place at time T/4. Furthermore, if we would have included viscous friction, the rope would almost be at rest at T/2 anyway.

In order to include viscous friction, we consider again the equations (A4, A5) for one protection point. A generalization to n protection points is obvious. We have

$$x_2 - x_1 = -\frac{m_2 \ddot{x}_2}{k_2} = \frac{\sigma}{k_2}$$

$$x_1 = \frac{\sigma}{e_1 k_1}$$

Adopting the language of viscoelasticity theory an elastic stress σ has now been introduced. Note that we have to take into account the friction force at P_1 with a resulting stress σ/e_1.

A Maxwell model [1] can now be constructed by adding a viscosity term to both equations after differentiating them

$$\dot{x}_1 = \frac{\dot{\sigma}}{e_1 k_1} + \frac{\sigma}{e_1 \eta_1}$$

$$\dot{x}_2 - \dot{x}_1 = \frac{\dot{\sigma}}{k_2} + \frac{\sigma}{\eta_2}$$

Eliminating $\dot{x}_1$ from the last equation, one gets

$$\dot{x}_2 = \frac{\dot{\sigma}}{k_2} + \frac{\sigma}{\eta_2} + \frac{\dot{\sigma}}{e_1 k_1} + \frac{\sigma}{e_1 \eta_1} = \dot{\sigma}\left(\frac{1}{k_2} + \frac{1}{e_1 k_1}\right) + \sigma\left(\frac{1}{\eta_2} + \frac{1}{e_1 \eta_1}\right) = \frac{\dot{\sigma}}{k} + \frac{\sigma}{\eta}$$

from which the effective η and k can immediately be read off

$$\frac{1}{\eta} = \left(\frac{1}{\eta_2} + \frac{1}{e_1 \eta_1}\right) \quad \text{and} \quad \frac{1}{k} = \left(\frac{1}{k_2} + \frac{1}{e_1 k_1}\right)$$

Thus we have a Maxwell model with effective material parameters k and η which gives a good description of a climbing rope for short times. It is given by equation (10) as shown in [1].